Approach," *Proceedings of the American Control Conference* (Boston, MA), June 1991, pp. 1935, 1936.

[4] Bernstein, D. S., Haddad, W. M., Hyland, D. C., and Tyan, F., "Maximum Entropy-Type Lyapunov Functions for Robust Stability and Performance Analysis," *Proceedings of the American Control Conference* (Chicago, IL), June 1992, pp. 2639–2643; also *Sys. Contr. Lett.*, Vol. 21, 1993, pp. 73–87.

[5] Seinfeld, D. R., Haddad, W. M., Bernstein, D. S., and Nett, C. N., "H_2/H_∞ Controller Synthesis: Illustrative Numerical Results via quasi-Newton Methods," *Proceedings of the American Control Conference* (Boston, MA), June 1991, pp. 1155, 1156.

[6] Collins, Jr., E. G., Davis, L. D., and Richter, S., "A Homotopy Algorithm for Maximum Entropy Design Lyapunov Equations," *Proceedings of the American Control Conference* (San Francisco, CA), June 1993, pp. 1010–1014.

Multiobjective Controller Design Using Eigenstructure Assignment and the Method of Inequalities

R. J. Patton,* G. P. Liu,† and J. Chen†
University of York,
York YO1 5DD, England, United Kingdom

Introduction

IN recent years, eigenstructure assignment has been an active topic of research in multivariable control theory. Since the degrees of freedom are available over and above pole assignment using state or output feedback,[1] respectively, numerous researchers have exercised those degrees of freedom to make the systems have good insensitivity to perturbations in the system parameter matrices via eigenstructure assignment.[1-5] Most eigenstructure assignment techniques in the last decade only pay attention to the optimal solutions for some special performance indices, e.g., $\|V_R\|_2\|V_R^{-1}\|_2$, where V_R is the right eigenvector matrix. However, many practical control systems are required to have the ability to satisfy simultaneously different and often conflicting performance criteria, for instance, closed-loop stability, low feedback gains, and insensitivity to model parameter variations.

In this Note, we provide a new approach to make the closed-loop system satisfy a set of required performance criteria with less conservatism, using eigenstructure assignment and the method of inequalities.[6]

Multiobjective Controller Design

Consider a linear multivariable time-invariant, completely controllable, state feedback system:

$$\dot{x} = Ax + Bu, \qquad u = Kx \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $K \in \mathbb{R}^{m\times n}$. Then the closed-loop system representation is given by $\dot{x} = (A + BK)x$. Introducing an $n \times n$-dimensional eigenvector matrix $V_R = [V_{R1}, V_{R2}, \ldots, V_{Rn}]$, where V_{Ri} $(i = 1, 2, \ldots, n)$ is the right eigenvector corresponding to the eigenvalue λ_i, a general solution for this problem can be given in the form of a parametric expression, $K(\Lambda, V_R)$, for all feedback gain matrices K which assign the self-conjugate set of eigenvalues $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ to the closed-loop system. If both the vector Λ and the right eigenvector matrix V_R of $(A + BK)$ are specified, the controller K is determined.

In practice, it is usually intended to locate the eigenvalue vector Λ in a well-defined set to meet the requirements of the practical control system (e.g., stability, speed of response, etc.). This leads to eigenvalue constraints, for example of the form $\lambda_{L_i} \le \lambda_i \le \lambda_{U_i}$, where $\lambda_{L_i} \in \mathbb{R}$ and $\lambda_{U_i} \in \mathbb{R}$ are the lower bound vector and the upper bound vector, respectively. These constraints may be removed by considering the change of variables given by

$$\lambda_i(\nu_i) = \lambda_{L_i} + (\lambda_{U_i} - \lambda_{L_i})\sin^2(\nu_i), \tag{2}$$

with $\nu_i \in \mathbb{R}$. Since the system is assumed to be completely controllable, the ith closed right eigenvector V_{Ri} is given by

$$V_{Ri} = (\lambda_i I - A)^{-1}BW_i, \qquad i = 1, 2, \ldots, n \tag{3}$$

where $W_i \in \mathbb{R}^{m\times 1}$ and the matrix $W = [W_1, W_2, \ldots, W_n]$.

For the case when the matrix $(\lambda_i I - A)$ is not invertible, for example when one or more closed-loop eigenvalues are required to be identical to open-loop values, then the following alternative to Eq. (3) by Roppenecker[5] and Liu and Patton[7] can be used without loss of generality. Clearly, the right eigenvector matrix V_R is a function of $\Upsilon = [\nu_1, \nu_2, \ldots, \nu_n]$ and W, i.e., $V_R(\Upsilon, W)$. Thus, the parametric formula of the controller matrix K can be described by $K(\Upsilon, W)$. A parametric representation of the control matrix K is given by[5]

$$K(\Upsilon, W) = WV_R^{-1}(\Upsilon, W) \tag{4}$$

In most parameter insensitive design methods using eigenstructure assignment, the performance indices are given on the basis of the right eigenvector matrix. For example, a very common performance index is given by

$$\phi(\Upsilon, W) = \|V_R\|_2\|V_R^{-1}\|_2 \tag{5}$$

where $\|V_R\|_2 = (\text{maximum eigenvalue of } V_R^* V_R)^{1/2}$.

Though the performance index $\phi(\Upsilon, W)$ can be used to represent an upper bound of the eigenvalue sensitivities, it is often conservative because of the following relations:

$$\phi(\Upsilon, W) \ge \max\{\phi_i(\Upsilon, W) : i \in \{1, 2, \ldots, n\}\} \tag{6}$$

where $\phi_i(\Upsilon, W)$ is the individual sensitivity of the eigenvalue λ_i to perturbations in any of the elements of the matrices A and B, defined by

$$\phi_i^2(\Upsilon, W) = \frac{(V_{Ri}^* V_{Ri})(V_{Li}^* V_{Li})}{(V_{Li}^T V_{Ri})^*(V_{Li}^T V_{Ri})}, \quad i = 1, 2, \ldots, n \tag{7}$$

where the superscript * denotes "conjugate-transposed," V_{Li} is the ith closed-loop left eigenvector given by the relation $V_L^T = V_R^{-1}$ with the left eigenvector matrix $V_L = [V_{L1}, V_{L2}, \ldots, V_{Ln}]$. Hence, to reduce the conservatism the problem becomes to find a pair (Υ, W) such that

$$\min_{\Upsilon, W}\{\phi_i(\Upsilon, W)\} \qquad \text{for } i = 1, 2, \ldots, n \tag{8}$$

To give a feel for the usefulness of the multiobjective approach as opposed to single-objective design techniques, let us consider the minimization of the cost functions $\phi_i(\Upsilon, W)$ $(i = 1, 2, \ldots, n)$. Let the minimum value of ϕ_i be given by ϕ_i^*, for $i = 1, 2, \ldots, n$, respectively. For these optimal values ϕ_i^*, there exist corresponding values given by $\phi_j(\phi_i^*)$ $(j \neq i,$

Received May 10, 1993; presented as Paper 93-3802 at the AIAA Guidance, Navigation, and Control Conference, Monterey, CA, Aug. 9–11; revision received Sept. 8, 1993; accepted for publication Sept. 10, 1993.

*Senior Lecturer, Department of Electronics. Senior Member AIAA.
†Research Fellow, Department of Electronics.

$j = 1, 2, \ldots, n$), for $i = 1, 2, \ldots, n$, respectively, and the following relations:

$$\min\{\phi_1(\phi_2^*), \phi_1(\phi_3^*), \ldots, \phi_1(\phi_n^*)\} \geq \phi_1^* \quad (9)$$

$$\min\{\phi_2(\phi_1^*), \phi_2(\phi_3^*), \ldots, \phi_2(\phi_n^*)\} \geq \phi_2^* \quad (10)$$

$$\vdots$$

$$\min\{\phi_n(\phi_1^*), \phi_n(\phi_2^*), \ldots, \phi_n(\phi_{n-1}^*)\} \geq \phi_n^* \quad (11)$$

If one of the performance functions $\phi_i(i = 1, 2, \ldots, n)$ is minimized individually (single-objective approach), then unacceptably large values may result for other performance functions $\phi_j(j \neq i, j = 1, 2, \ldots, n)$. The single- (or mixed) objective approach has been considered in earlier papers.[3,4] Generally, there does not exist a solution for Eq. (8). So, we reformulate the problem into a multiobjective problem as

$$\phi_i(\mathbf{Y}, \boldsymbol{W}) \leq \varepsilon_i, \qquad \text{for } i = 1, 2, \ldots, n \quad (12)$$

where $\varepsilon_i \in [\min\{\phi_i(\mathbf{Y}, \boldsymbol{W})\}, \min\{\phi(\mathbf{Y}, \boldsymbol{W})\}]$ is determined by the designers.

Clearly, if we let all $\varepsilon_i = \min\{\phi(\mathbf{Y}, \boldsymbol{W})\}$ for $i = 1, 2, \ldots, n$, the design problem is the same as the minimization of the performance function $\phi(\mathbf{Y}, \boldsymbol{W})$, which is a conservative formulation. On the other hand, if we let all $\varepsilon_i = \min\{\phi_i(\mathbf{Y}, \boldsymbol{W})\}$ for $i = 1, 2, \ldots, n$, it is the same as Eq. (8), which is a radical formulation probably without solution. Hence, according to the practical problem ε_i should be adjusted so that the design formulation is neither conservative like the former nor radical like the latter. In addition, we often need to consider some constraints on the controller gain matrix $\boldsymbol{K}$. A scalar measure criterion which quantifies a structurally constrained gain matrix may be defined as follows

$$\phi_{n+1}(\mathbf{Y}, \boldsymbol{W}) = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}\boldsymbol{K}_{ij}^2(\mathbf{Y}, \boldsymbol{W})\right]^{1/2} \leq \varepsilon_{n+1} \quad (13)$$

where $\boldsymbol{K}_{ij}(\mathbf{Y}, \boldsymbol{W})$ is the (i, j)th element of the full state gain matrix $\boldsymbol{K}$, β_{ij} is a positive weighting factor which may be used to force certain elements of the gain matrix to become small, and ε_{n+1} is a positive number which is determined by the designers.

Therefore, combining Eqs. (12) and (13) results in the following control problem:

$$\phi_i(\mathbf{Y}, \boldsymbol{W}) \leq \varepsilon_i, \qquad \text{for } i = 1, 2, \ldots, n, n+1 \quad (14)$$

For the output feedback case, some elements of the gain matrix $\boldsymbol{K}$ are zero-valued and the corresponding weighting parameters β_{ij} can be selected to have large values to force this structure.[3] There is therefore no loss of generality in using the formulation given here for either state or output feedback control problems.

Based on the method of inequalities,[6,8,9] a numerical solution for a controller satisfying the performance criteria, Eq. (14), can be obtained. Let $\boldsymbol{\Gamma}_i$ be the set of parameters $(\mathbf{Y}, \boldsymbol{W})$ for which the ith performance criterion is satisfied:

$$\boldsymbol{\Gamma}_i = \{(\mathbf{Y}, \boldsymbol{W}) : \phi_i(\mathbf{Y}, \boldsymbol{W}) \leq \varepsilon_i\} \quad (15)$$

Then the admissible or feasible set of parameters for which all of the performance criteria hold is the intersection

$$\boldsymbol{\Gamma} = \bigcap_{i=1}^{n+1} \boldsymbol{\Gamma}_i \quad (16)$$

The search for an admissible pair $(\mathbf{Y}, \boldsymbol{W})$ can be pursued by solving

$$\min_{\mathbf{Y},\boldsymbol{W}}\{\max\{\phi_i(\mathbf{Y}, \boldsymbol{W}) - \varepsilon_i : i = 1, 2, \ldots, n+1\}\} \leq 0 \quad (17)$$

During the above minimization, if one or more of the $\phi_i(\mathbf{Y}, \boldsymbol{W})$ persists in being larger than ε_i, this may be taken as an indication that the performance criteria may be inconsistent, whilst their magnitude gives some measure of how closely it is possible to approach the objectives. In this case, some of the performance criteria should be relaxed until they are satisfied. From a practical viewpoint, the approximate optimal solution is also useful if the optimal solution is not achievable. We do not discuss the method of inequalities in this Note. However, a full discussion of this method is given in Zakian and Al-Naib.[6]

Example

Consider the linear equation of motion of lateral dynamics of the L-1011 aircraft corresponding to a certain cruise flight condition. The system matrices for this example are given by[10]

$$\boldsymbol{A} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.8900 & 0.3900 & -5.5550 \\ 0.0000 & -0.0340 & -2.9800 & 2.4300 \\ 0.0350 & -0.0011 & -0.9900 & -0.2100 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.000 & 0.000 \\ 0.760 & -1.600 \\ -0.950 & -0.032 \\ 0.030 & 0.000 \end{bmatrix}$$

The closed-loop eigenvalues are required to lie in ranges given by

$$\lambda_1 \in [-1.5, -0.5], \qquad \lambda_2 \in [-2.5, -1.5]$$

$$\lambda_3 \in [-3.5, -2.5], \qquad \lambda_4 \in [-4.5, -3.5]$$

A state feedback controller is required to assign the eigenvalues in the above ranges to satisfy the following performance criteria:

$$[\phi_1(\mathbf{Y}, \boldsymbol{W}), \phi_2(\mathbf{Y}, \boldsymbol{W}), \phi_3(\mathbf{Y}, \boldsymbol{W}), \phi_4(\mathbf{Y}, \boldsymbol{W}), \phi_5(\mathbf{Y}, \boldsymbol{W})] \leq [2.3, 4.1, 2.3, 4.1, 9.0]$$

The performance function ϕ_i is the sensitivity of each eigenvalue λ_i, for $i = 1, 2, 3, 4$, respectively. The performance function ϕ_5 is the gain matrix measure, which is defined as Eq. (13) with $\beta_{ij} = 1$, for $i = 1, 2, 3, 4$ and $j = 1, 2$. Using the method of inequalities, this has led to the following design results:

$$\mathbf{Y} = \begin{bmatrix} 0.2000 \\ 0.3000 \\ -0.3702 \\ 2.118 \end{bmatrix}^T$$

$$\boldsymbol{W} = \begin{bmatrix} 0.7200 & -1.5148 \\ -1.3981 & 4.7985 \\ 2.2362 & -3.8949 \\ 6.4527 & 3.1259 \end{bmatrix}^T$$

$$\begin{bmatrix} \phi_1(\mathbf{Y}, \boldsymbol{W}) \\ \phi_2(\mathbf{Y}, \boldsymbol{W}) \\ \phi_3(\mathbf{Y}, \boldsymbol{W}) \\ \phi_4(\mathbf{Y}, \boldsymbol{W}) \\ \phi_5(\mathbf{Y}, \boldsymbol{W}) \end{bmatrix} = \begin{bmatrix} 2.2378 \\ 4.0807 \\ 2.2326 \\ 4.0893 \\ 8.7863 \end{bmatrix}$$

Substituting $\mathbf{Y}$ and $\mathbf{W}$ into Eqs. (2–5) gives the closed-loop eigenvalues, eigenvectors, the controller gain matrix, and the function $\phi(\mathbf{Y}, \mathbf{W})$ as follows:

$$\Lambda = [-1.0184 \quad -2.0165 \quad -3.0148 \quad -3.9908]$$

$$\phi(\mathbf{Y}, \mathbf{W}) = 8.0840$$

$$V_R = \begin{bmatrix} 1.9593 & 0.7860 & 1.3913 & -0.4749 \\ -1.9954 & -1.5850 & -4.1943 & 1.8951 \\ 0.8334 & -3.3813 & 2.1977 & 3.8898 \\ 0.9064 & -1.8470 & 0.7328 & 0.9723 \end{bmatrix}$$

$$K = \begin{bmatrix} 1.6569 & 3.2205 \\ 0.6663 & 1.8746 \\ 2.4237 & 1.8304 \\ -3.5491 & -6.1889 \end{bmatrix}^T$$

From the above, it can be seen that the performance function $\phi(\mathbf{Y}, \mathbf{W})$ (8.0840) is much larger than the maximum (4.0893) of the individual eigenvalue sensitivities $\phi_i(\mathbf{Y}, \mathbf{W})$ $(i = 1, 2, 3, 4)$ which have met the design requirements. This indicates that the individual eigenvalue sensitivity functions are more accurate and better to describe the system insensitivity property to perturbations than the performance function $\phi(\mathbf{Y}, \mathbf{W})$. If the gain matrix elements were considered to be too large, a further reduction would be possible by sacrificing parameter insensitivity (and vice versa), i.e., the current level of one or more individual eigenvalue sensitivities, by appropriate adjustments to ε_i $(i = 1, 2, 3, 4)$. In this way, we can design the controller to meet the specific needs.

Conclusions

A parameter-insensitive design method using eigenstructure assignment and the method of inequalities has been developed for multivariable control systems. Based on the eigenstructure assignment principle, the control problem is formulated as that of finding a controller to satisfy a set of performance criteria described by a set of inequalities, which includes requirements of individual eigenvalue sensitivities and feedback gains. The performance criteria describe the practical control problem more accurately and less conservatively than would be possible using scalar performance criterion approaches. A numerical solution has been presented using the method of inequalities and illustrated in the design of an aircraft system controller.

References

[1] Moore, B. C., "On the Flexibility Offered by State Feedback in Multivariable Systems Beyond Closed-Loop Eigenvalue Assignment," *IEEE Transactions on Automatic Control*, Vol. 21, No. 6, 1976, pp. 689–692.

[2] Burrows, S. P., and Patton, R. J., "A Comparison of Some Robust Eigenvalue Assignment Techniques," *Optimal Control Application and Methods*, Vol. 11, No. 4, 1990, pp. 355–362.

[3] Burrows, S. P., and Patton, R. J., "Design of a Low Sensitivity, Minimum Norm and Structurally Constrained Control Law Using Eigenstructure Assignment," *Optimal Control Application and Methods*, Vol. 12, No. 3, 1991, pp. 131–140.

[4] Kautsky, J., Nichols, N. K., and Dooren, P. Van., "Robust Pole Assignment in Linear State Feedback," *International Journal of Control*, Vol. 41, No. 5, 1985, pp. 1129–1155.

[5] Roppenecker, G., "On Parametric State Feedback Design," *International Journal of Control*, Vol. 43, No. 3, 1986, pp. 793–804.

[6] Zakian, V., and Al-Naib, U., "Design of Dynamical and Control Systems by the Method of Inequalities," *Proceedings of the Institute of Electrical Engineering*, Vol. 120, No. 11, 1973, pp. 1421–1427.

[7] Liu, G. P., and Patton, R. J., "Parametric State Feedback Design of Multivariable Control Systems Using Eigenstructure Assignment," *Proc. of the 32nd IEEE Conference on Decision and Control*, TX, Dec. 1993, 835-836.

[8] Liu, G. P., "Theory and Design of Critical Control Systems," Ph.D. Thesis, Control Systems Centre, Univ. of Manchester Inst. of Science and Technology, Manchester, England, UK, Feb. 1992.

[9] Whidborne, J. F., and Liu, G. P., *Critical Control Systems: Theory, Design and Applications*, Research Studies Press Limited, UK, 1993.

[10] Andry, A. E., Chung, J. C., and Shapiro, E. Y., "Modalized Observers," *IEEE Transactions on Automatic Control*, Vol. 7, No. 7, 1984, pp. 669–672.

Component Model Reduction Methodology for Articulated Multiflexible Body Structures

Allan Y. Lee* and Walter S. Tsuha†
Jet Propulsion Laboratory,
California Institute of Technology,
Pasadena, California 91109

Introduction

TO simulate and analyze the dynamical motion of an articulated, multiflexible body structure, we can use multibody simulation packages such as Dynamic Interaction Simulation Controls and Structure (DISCOS).[1] To this end, we must supply appropriate reduced-order models for all the flexible components involved. For complex systems, such as the Galileo spacecraft, practical considerations (e.g., simulation time) impose limits on the number of modes that each flexible body can retain in a given simulation. Hence, reduced-order models of the system's flexible components are needed.

Model reduction methodologies typically are used to reduce a large system model to one that is small enough to facilitate analysis and control design, yet "rich" enough that it retains the salient features of the original system model. Although the literature on model reduction is vast (see, e.g., Ref. 2), works that address the model reduction needs of articulated, multiflexible body structures are sparse, and have appeared only recently.[3–6] Here, what is needed is a way of generating reduced-order component models, which, when reassembled, produce a reduced-order system model that retains the important input-to-output mapping of the original system. The enhanced projection and assembly[6] technique is one way to perform this task.

In this method, a composite mode set, consisting of important system modes from all system configurations of interest, not just from one particular system configuration, is first selected. It is then augmented with static correction modes before being projected onto the component models to generate reduced-order component models. To generate the composite mode set, eigenvalue problems for the full-order system models, at all configurations of interest, must be solved repetitively. This is a drawback of the method because solving large eigenvalue problems can be costly. To overcome this difficulty, a two-stage model reduction methodology, combining the classic component mode synthesis method and the enhanced projection and assembly method, is proposed in this research (Fig. 1).

Received Feb. 5, 1993; revision Oct. 8, 1993; accepted for publication Oct. 15, 1993.

*Mail Stop 198-326, Member of Technical Staff, Guidance and Control Section, 4800 Oak Grove Drive.

†Mail Stop 157-410, Member of Technical Staff, Communications Systems Research Section, 4800 Oak Grove Drive.